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Negative Interest Rates and Their Technical Consequences
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1. Introduction

The economic world today is different from what we have seen before in history, and nobody dared to expect what is reality now: negative nominal interest rates for long-term maturities spread over large parts of the world. These negative interest rates have tremendous economic effects, but beside this they also cause a lot of technical problems. These problems range from the inability of spreadsheet functions to deal with negative inputs to the need to adapt mathematical models built for creating sensible sets of future scenarios. These technical problems can lead to questioning of model results and could eventually result in a situation where the financial position of parts of the financial industry cannot be valued and hence cannot be managed effectively.

The Actuarial Association of Europe as the European actuarial professional body including experts in risk management and quantitative modelling started a Task Force to describe the current situation and provide possible solutions to overcome this critical issue. The Task Force consists of Malcolm Kemp, Philipp Keller, Christoph Krischanitz (chair) and Bertrand Lespinasse, and was initiated and steered by Karel Goossens.

In chapter 2 we discuss the main approaches to model negative interest rates. These approaches depend on an assumption about whether there exists a floor bigger than -100% for negative interest rates. We therefore also discuss the possible existence of such a floor and what are the current economic estimates for this floor. We do this in chapter 3. Chapter 4 discusses briefly some ideas for further areas which are affected by negative interest rates and we conclude the paper with references and an appendix with some technical details.

2. Modelling negative interest rates

Negative nominal interest rates ('IR') currently present in IR markets are causing some insurance undertakings to question their IR modelling tools. This questioning encompasses both the financial models embedded in their economic scenario generators ('ESGs') (either 'risk-neutral' or 'real-world' variants), as well as the formulae included in their ALM tools.

Impact on valuation formulae included in ALM tools

ALM models are heavily dependent on suitable ways of valuing assets and liabilities and so need valuation formulae that behave properly when being fed with negative interest rates.
Traditional formulae used to value bonds and more generally used to discount cash-flows are still relevant in a negative interest rate environment. However, this is not so true for the valuation of interest rate derivatives. Many continental European insurers have traditionally focused on using the 1976 Black Model for valuing swaptions, caps and floors in ALM models. In its basic form, this model

(which models evolution of interest rates using a log-normal process) assumes interest rates are positive.
The solution generally now being adopted is to use a refinement that includes a shift or displacement term (named Θ, being a positive value), which defines the opposite of the lowest value it is possible for forward rates to exhibit. Valuation formulae are essentially unchanged, provided a suitable translation adjustment is applied to the strike and the forward rate. This approach is called using a Displaced Diffusion ('DD') model and e.g. values a call option in the following manner (where 'B76' identifies the corresponding formula applicable under the basic 1976 Black Model):

$$C_{DD}(T, K, F_t) = C_{B76}\left(T, K + \Theta, F_t + \Theta, \sigma_{impl}^{DD}(T, K + \Theta)\right)$$

It should be noted that even though the formula remains essentially unchanged, the implied volatility input on which it depends needs to be re-estimated, as the meaning of this parameter depends on the displacement involved.

Selection of the interest rate model (both risk-neutral and real-world)

The zero-coupon ('ZC') curve provided by EIOPA that forms the starting point for Solvency II Pillar I calculations has regularly displayed negative rates for some maturities and some currencies in recent months. To reflect this new economic situation, the IR diffusion models used in ESG need to allow for negative rates.
Thus, models such as the Black Karasinski and LIBOR Market Model, that model forward and ZC rates in a strictly positive fashion, are no longer appropriate (particularly for risk-neutral scenarios) and alternative models should be considered. Criteria usually used to select interest rate models encompass the following:

- Ability of the model to reproduce the initial ZC curve,
- Ability to reproduce volatility surfaces and smiles,
- Ability to price "out of calibration sample" implied volatilities (i.e. implied volatilities of assets that were not part of the calibration basket),
- Good balance between adequate number of parameters and over-parameterization.

For risk-neutral computations, various models appear relevant in the sense that they exhibit some or all the above criteria:

(1) Some models that involve diffusions of normally distributed instantaneous rates permit simulation of negative rates while also exactly reproducing the initial ZC curve. Examples include the standard 1-factor Hull-White model and the G2++ model (equivalent to a 2-factor Hull-White model). The 1-factor Hull-White model includes only one risk factor; it can therefore model interest rate level translation risk well but not slope change risk. Multi-factor models can cater for both types of evolution. Empirically, the 1-factor Hull-White model also contains too few parameters to properly replicate observed implied volatilities. In contrast, the G2++ seems to reproduce at-the-money (ATM) volatilities well, but does not appear to be fully satisfactory when attempting to replicate out-of-the-money (OTM) volatilities, and therefore to properly allow for volatility smiles.

(2) Some models adapted from the forward diffusion LIBOR Market Model also appear to be particularly robust. Indeed, they provide negative interest rates, fit the initial ZC curve, while also fitting volatility surfaces and smiles well. Such models include the Displaced Diffusion – LIBOR Market Model ('DD-LMM') and the LMM+ (which is an adaptation of the DD-LMM with a stochastic volatility and therefore potentially offering an even better fit to observed volatility smiles).

(3) The shifted Cox-Ingersoll-Ross (CIR++) and CIR2++ (2-factor CIR++) are based on a shifted χ^2 diffusion for the instantaneous rate and they can also reproduce volatility smiles.

When modelling 'real-world' scenarios and when trying to identify suitable stresses to apply to interest rates in an internal model it is not so crucial that the model used satisfies the martingale property (for a discussion of the martingale property see below). Instead, commonly the model might rely on time-series approaches (such as principal components analysis applied to past data) encompassing major risk factors. Models that cater for periods when interest rates are higher as well as periods when they are lower may characterise the world as exhibiting two (or possibly more) states of the world which are assumed to exhibit similar characteristics other than having a higher or lower general level of interest rates. Shocks can then be identified contingent on the state of the world at the time the stress is assumed to arise.

Integration of a lowest absolute level in the IR diffusion

Several of the models described above need as an input a lowest absolute level to which interest rates may fall. Assuming that such an absolute lowest level can be identified (see Chapter 2), the identified level should be cautiously introduced in the scenario generation process, in the sense that this absolute lowest level should have as little as possible impact on the *Martingality* (i.e. the ability of the scenarios to exhibit the martingale property and hence deliver arbitrage-free convergent Monte Carlo estimators for any set of cash-flows) and hence on the *Market Consistency* (i.e. the ability to reproduce prices and hence implied volatilities) of the scenario set. In general, for a scenario set to provide a valid way of valuing cash flows under Solvency II it needs to adhere to the martingale property.

For Gaussian models, imposing any lower absolute level of interest rates can only be done *a posteriori*, through the application of a floor level below which any simulated rate is then pulled up. The issue with such an approach is that it truncates part of the simulation set. As a consequence, it affects the properties of economic scenario set as far as martingality and market-consistency are concerned.

For "displaced" or "shifted" models such as the CIR++, CIR2++, DD-LMM and LMM+, we can of course also *a posteriori* floor the interest rates (with the same impact on martingality and market-consistency). However, one parameter of the model can *a priori* ensure that the IR scenarios cannot cross a fence, without disrupting the ability of the model to provide market consistent estimates. This is the shift parameter. The scenarios are structurally conditioned not to be fall below the shift's opposite value (i.e. $-\Theta$ % if the chosen shift is $\Theta\%>0$). As no post-generation treatment is then carried out, the use of such a shift does not invalidate the martingality of the scenarios used.

Inclusion of a shift parameter does not necessarily make the model calibration (much) more onerous. The shift parameter does not have to be calibrated along with the other model parameters at each valuation date, but can instead be considered a meta-parameter not necessarily updated for each calibration. In this case, the meta-parameter is determined outside the regular calibration process, probably after carrying out intensive traditional calibration processes (e.g. minimization of the squared differences between swaptions market prices and model prices) in which the shift parameter is indeed considered as a parameter. Intensive calibration processes are desirable in the sense that multiple initial values for the set of parameters as well as multiple initial economic conditions should be tested to avoid erroneously selecting a local optimum when a better global optimum exists. When determining the shift to be used, it should be noted that the lower the Θ shift the less negative interest rates can become. It is also worth noting that replication of initial implied volatilities surfaces may be hindered by adopting small values of Θ. A consequence is that interest rates models such as DD-LMM and the LMM+ can produce explosive IR scenarios when volatilities are high together with a low shift value. These explosive interest rates situations may not be accepted by ALM models. An IR *cap* could be applied to temper these explosive scenarios but with the same concerns regarding martingality as the ones raises with the aforementioned *floor* applied to the generated scenarios, if the cap is introduced in an *a posteriori* manner. It is in principle possible to introduce caps in an *a priori* manner, but we are not aware of such modelling being done in practice by market participants.

Normal or lognormal implied volatilities?

Two kinds of paradigms are used to translate observed option prices into implied volatilities, depending on the assumed swap rate model underlying the framework in which the volatility is denominated.

First it is possible to assume that swap rates follow a lognormally distributed process (the 'Black model'). This paradigm raises an issue when the swap rate is very low because this 'lognormal' volatility scales in line with the inverse of the swap rate. So, when swap rates are very low, the lognormal implied volatility can explode. Moreover, when the swap rate is negative, the lognormal implied volatility becomes ill-defined (unless a shift term has been introduced as above).
Second, it is possible to assume that swap rate follow a normally distributed process (the 'Bachelier model'). This second paradigm appears to be more stable since it enables full implied volatilities matrices to be generated at any time, whatever the level of the swap rate. In practice, this 'normal' implied volatility is more independent of the level of the swap rates and more stable through time than a 'lognormal' implied volatility (see Figure 1 below).

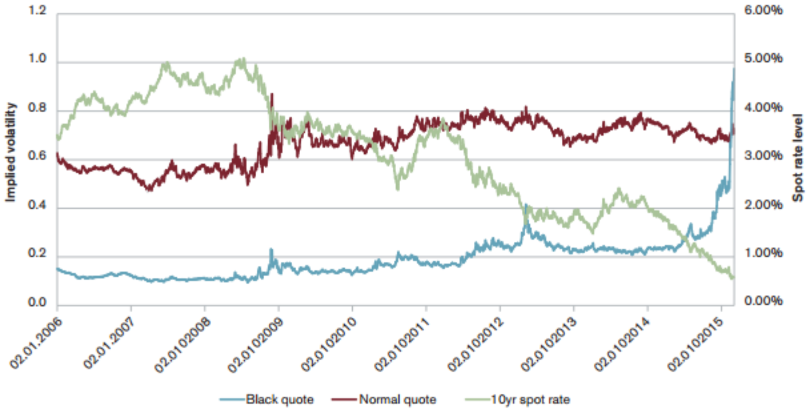

Figure 1: implied volatilities vs. 10y spot rate 2006-2015. Source: Milliman (09.2015) - The new Normal

The choice of the implied volatility paradigm as above is formally independent of the choice of the underlying interest rate diffusion model. This is because the choice is driven by market conventions on how prices for different option instruments are quoted, and hence usually needs adapting to be consistent with input parameters actually applicable to whatever model the user considers applies to the future evolution of interest rates. Closed-form formulae to derive swaption implied volatilities exist under both of the above paradigms.

What are the consequences of not modelling negative interest rates?

Failure in the current economic situation to model the potential for interest rates to become (more) negative leads to problems in Solvency II Best Estimate and Standard Formula calculations. These problems depend on the underlying interest rate modelling approach:

(A) If the interest rate model used assumes positive interest rates (such as LMM model or Black Karasinski), then:

   (a) It becomes impossible to reproduce the actual spot interest rate curve.

   (b) The Best Estimate of Technical Provisions calculated using the scenario set generated by the ESG would then underestimate the cost of financial options and guarantees.

   (c) In particular, the ESG can fail to work effectively in situations where guaranteed rates embedded in savings contracts (such as those sold by some Continental life insurers) applied. Such a model would, however, allow the production of sets of economic scenarios that display satisfactory martingale tests and appropriate level of implied volatilities for OTM options with high enough positive strikes.

(B) If the interest rate model encompasses negative interest rates (e.g. LMM+, G2++) but if a 0-floor (or a floor above the practical lower level to which interest rates might fall) is applied *a posteriori* to the generated scenarios, then the situation would be as follows:
   (a) As in the case where the interest rates model assumes positive interest rates, the cost of financial options and guarantees could also be underestimated, and the discount factors used to discount the cash-flow part of the Best Estimate would be artificially high, leading a underestimation of the Best Estimate.
   (b) In addition, the martingality of economic scenarios would be broken and as a consequence, estimators of the Best Estimate of Technical Provisions would not converge to the correct answer; this deterioration in the estimators' convergence could then result in an inflated capital position.

3. Negative interest rates: How low can they go?

When shift models are used, it becomes natural to ask how such a shift can be assessed. In other words, we need to identify whether it is appropriate to assume that there is a floor for negative interest rates bigger than -100%. It is difficult to get a final answer on this question, but there are several arguments and estimations available which are worthwhile to consider which we set out below.

The current negative interest rate environment is unprecedented. Japanese banks had to pay negative rates for short term deposits at the end of 1998 in the inter-bank market. In the early 1970s, the CHF appreciated against the USD. To counteract this, the Swiss National Bank introduced levies on Swiss franc deposits by non-residents, starting with a 2% quarterly charge in 1972 and increasing to 10% in 1978. This was however unsuccessful and the CHF strengthened from 4.33 CHF for 1 USD to 1.47 USD in 1978. However, these were singular episodes and not part of a general trend[1].

Since the financial crisis, negative interest rates are being used extensively. Since the beginning of the current financial crisis in 2008, several central banks have introduced negative interest rate policies (NIRPs). The Swedish Riksbank lowered its deposit rate below zero in 2009. In 2012, the Danish National Bank introduced negative rates due to the appreciation of the DKR during the European sovereign debt crisis. Currently interest rates set by the following central banks are also in negative territory: the Bank of Japan (BoJ), the Czech National Bank (CNB), the European Central Bank (ECB), the Central Bank of Hungary (MNB), the Norwegian Central Bank (NCB), and the Swiss National Bank (SNB)[2]. As of September 2016, government debt with negative rates totalled USD 10.9tn out of a total of about USD 150tn sovereign debt outstanding globally. Roughly half of EUR government bonds and about 33% of industrialized countries sovereign bonds issued have a negative nominal yield[3].

There are several reasons for imposing negative interest rates. These include financial repression, weakening the currency and fighting against deflation. Some central banks have used negative rates to counter inflation or to give incentives to borrow and invest (ECB, BoJ, SR), others were forced to adopt NIRPs by the appreciation of their currencies and to mitigate spill-over effects from unconventional monetary policy measures of other jurisdictions (DNB, SNB).

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[1] Monetary Policy Under Low Interest Rates: The Experience of Switzerland in the late 1970s, Peter Kruger & Georg Rich, February 2001

[2] Negative Interest Rate Policy (NIRP): Implications for Monetary Transmission and Bank Profitability in the Euro Area, IMF Working Paper WP/16/172, Andreas Jobst & Huidan Lin, August 2016

[3] QE Monitor: Dreaming a little taper dream, Allianz Global Investors, Ann-Katrin Petersen Global Capital Markets & Thematic Research 7 October 2016

Theory distinguishes between two lower bounds for the short-term nominal interest rate: the economic and the physical lower bound. The physical lower bound equals the opportunity costs of holding capital. The economic lower bound is given by cost the rates impose on the banking sector or where further rate cuts cease to provide stimulus to the economy.

The physical lower bound is given by the cost of storing money: If the cost of keeping money in a bank account is higher than the storage cost including insurance, it is rational to shift into cash. Estimates of these costs vary and range from 50bp to 200bp.
The economic lower bound is determined by the cost to the banking sector. Negative rates reduce profits to banks since they are often not able to pass through the costs to retail customers due to the fear of a run on the bank. In Switzerland, with short term rates at around -75bp, the cost to banks is estimated to be about 0.03% of total assets.

If interest rates drop below the physical lower bound, there is a high risk of households and businesses withdrawing their deposits. The large-scale hoarding of cash could lead to risks to financial stability and the possibility of runs on banks. The lower the cost of holding cash and the longer the period of negative rates is predicted to be, the higher is this risk. Central banks and governments can mitigate this risk by imposing an upper limit on cash transactions[4], moral suasion to business and banks not to turn reserves into cash[5], removing high denominated bank notes from circulation[6] and – as the ultimate measure – forbidding the use of cash[7] (by 'cash' we here mean actual bank notes and coins).

Central banks have several ways to depress nominal interest rates while limiting the risk of large-scale hoarding of cash. Central banks can use approaches which protect domestic retail depositors from negative rates. They can use a tiered-reserve regime to put the cost onto wholesale and foreign investors without retail and corporate customers being directly affected. They can exempt rates from negative rates depending on their participation in lending schemes that the central banks offer.

There are estimates that the ECB can depress the rate to -4.5% without negative rates being transmitted to retail customers. This estimate assumes that the ECB limits the cost to the banking sector to 0.03% of total assets and that it uses a tiered-reserve regime. The estimate is calibrated on the Swiss experience and assumes only modest changes to the reserve regime. For the USD, a rate of -1.3% is estimated and for the GBP of -2.5%[8].

If central banks were to abolish paper money, interest rates might become much more negative. The estimates on the lower limits of interest rates are ultimately based on bank customers not switching their deposits into cash. If physical money were not to exist anymore, there would obviously be no possibility of such a run occurring. The only limit for interest rate would then be the economic lower bound.

Even if central banks do not abolish cash, they could increase the cost of holding physical currency. A simple way of increasing the cost and lowering the physical lower bound is to remove high denomination notes from circulation. Central banks justify this often with the argument that these notes are used mostly by criminals. However, the most valuable bank note in circulation - Swiss CHF 1000 – is used mainly as a store of value, according to the SNB. All things equal, the physical lower bound becomes lower as the value of the largest bank note that is in circulation falls. Another

[4] HSBC imposes restrictions on large cash withdrawals, Bob Howard, BBC, 24 January 2014
[5] Pensionskassen wollen Bargeld - Bank weigert sich, Handelszeitung, 12 March 2015
[6] ECB ends production and issuance of €500 banknote, ECB Press Release, 4 May 2016
[7] How far can the repo rate be cut? Jan Alsterlind, Hanna Armelius, David Forsman, Björn Jönsson and Anna-Lena Wretman, 11 November 2015, Sveriges Riksbank
[8] Negative policy rates: The bound is lower than you think, Special Report, J.P.Morgan, Malcolm Barr, Bruce Kasman & David Mackie, 9 February 2016

approach would be to introduce a tax on cash holdings (Gesell tax). This could be done by introducing money with an expiry date that has to be stamped and would be taxed if it were to remain legal tender, which would be a modern form of medieval concept of demurrage.

The economic lower bound depends on assumptions made by the central bank. In contrast to the hard physical lower bound which depends on the relatively easily determined cost of holding cash, the economic lower bound is more subjective. The ECB defines the economic lower bound as that level below which further rate cuts cease to provide aggregate stimulus to the economy[9]. This assessment is obviously dependent on many assumptions. It is not clear that the estimated cost currently imposed on Swiss banks (0.03% of total assets) is an upper bound. Central banks – in particular, in jurisdictions where large banks effectively rely on government support – might consider the benefit to the wider economy higher than costs to the banking sector.

Historical data offers little guidance to the lower limit that nominal interest rates can take. The current situation is unprecedented. What little data is available from past episodes of negative nominal interest rates (Japan and Switzerland) is likely not very relevant for a situation where over USD 10tn of outstanding debt has a negative yield. Scenario analysis and macroeconomic models that aim to predict central bank policies may be more useful to gain insight into possible future interest rates.

There are scenarios imaginable where interest rates are lower than today. Examples of potential scenarios to consider are:

- The ECB abolishing paper money. The deteriorating solvency of European banks leads to the risk of a run on banks in its jurisdiction. The ECB decides to mitigate the risk by abolishing paper money. This allows the ECB to push rates down to unprecedented levels without fear of
- An appreciation of the EUR against other currencies, which leads to a reassessment of the ECB of the economic lower bound. Protectionist policies by the US lead to a trade war with Asia, and uncertainty regarding the US economy. The USD drops and the EUR is considered a safe-haven, leading to its appreciation. Exports collapse and the ECB lowers the interest rates of the EUR to depreciate it.
- A currency war between the EU and the USA. The US government aims to push its export industry by depreciating the USD. The ECB follows suit and the EUR and USD rates enter into a downward spiral.
- Global recession. Protectionist measures by the US lead to a trade war and a steep deflationary recession. The ECB responds by lowering rates into deep negative territory.
- Regulatory flight to the bottom. The US dismantles Dodd-Franks and the BoE and EBA follow suit. Excessive rent seeking and risk taking by large banks lead to systemic collapse of the banking sector with widespread bailouts. Catastrophic government debt of EEA member states forces the ECB to keep the interest rates negative for the government to be able to finance their debt.

The abolition of cash is likely the single biggest risk to insurers that write long-term liabilities. The presence of cash imposes discipline on central banks and the banking sector. It defines a physical lower bound on nominal interest rate. The hoarding of cash is a signal to central banks and finance ministries on the population's assessment of their monetary and economic policies. Cash ensures the possibility to engage in private transaction, which is one of the foundations for the trust in an economy and in society. Without physical cash, nominal interest rates are limited only by a central bank's estimate of the economic lower bound, which is highly subjective. In a cashless economy, it is difficult to define any lower bound that interest rates can reach.

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[9] Assessing the implications of negative interest rates, Speech by Benoît Coeuré, Member of the Executive Board of the ECB, 28 July 2016

The Solvency II risk margin (and equivalents in other regulatory regimes) should reflect the uncertainty present in the lowest level that interest rates can take. Modelling approaches that include shifts as above (i.e. include parameters such as the Θ referred to above) typically rely on estimates of the lowest absolute values that interest rates can take. This estimate is uncertain and is not practically hedgeable. Arguably, this uncertainty **- as for example the uncertainty of the Ultimate Forward Rate, reinvestment risk and other assumptions**, should therefore be captured in the risk margin[10]. Note that this uncertainty should be captured over the entire duration of the insurance liabilities.

4. Wider implications of low or negative interest rates

Commentators differ on why interest rates in many European countries are low at present.
Some commentators believe that a major contributor has been 'unconventional' monetary policies introduced by many governments in response to the 2008 Global Financial Crisis. This line of reasoning highlights the desire of many governments to boost demand by lowering interest rates and thereby combat a weak economic environment. By products of such policies include:

(a) Monetary values of longer-term assets including bonds have risen, pulling down longer-term yields
(b) These valuation shifts have redistributed wealth within society, typically towards wealthy older individuals, as these individuals own a relatively high proportion of the existing stock of such assets
(c) Long-term guarantees such as those present in some life insurance policies and in many defined benefit pension fund benefit structures have become more expensive to honour. This has stressed the balance sheets of such institutions.
(d) The interest margin available to banks (i.e. the difference between the rates of interest they charge borrowers and the rates of interest they pay savers) has been squeezed, affecting their profitability.

Other commentators stress longer-term (multi-decade) trends that had been pushing down interest rates even prior to the Crisis and have continued to do so since then. This line of reasoning argues that shifting demographic profiles and other factors have increased the supply of savings and reduced the demand for borrowings. This line of reasoning argues that interest rates have fallen to address the supply/demand dynamic created by these factors.

There probably is some validity to both lines of argument. Whatever the ultimate driver of low interest rates, any interest rate exposure involves cash flows flowing from one party to another. For example, cash flows are paid by the issuer of a bond to the owner of that bond. This means that (unexpected) interest rate changes create redistributive effects, transferring wealth between different economic agents, see (b) above.

Very low interest rates lead to a transfer of wealth away from the insurance and pension industry. Low interest rates lead to some winners. These potentially include some governments if they can roll-over debt on favourable terms , which may allow them to postpone structural measures and to further increase government debt (if the main purchasers of such debt are not their own central banks). Indisputable losers of low interest rate policies are the insurance and pension industry. The increased cost to produce insurance liability cash flows and the low returns of bonds lead to deteriorating balance sheets and financial positions.

[10] Market Consistency, AAE (Groupe Consultatif) 2012

5. Appendix: interest rates diffusion models description

1. Shifted models

The models presented here are adapted from the lognormal forward diffusion LIBOR Market Model. Forwards are directly observable on markets, which is not the case for the instantaneous rates presented in section 2. We begin this section by recalling the LIBOR Market Model structure. This model is not shifted but the LMM subsection is very useful to understand the shifted models diffusions presented below.

1.0. LIBOR Market Model (LMM) [1]

1.0.1. Notation

- $P(t,T)$ is the ZC price of maturity $T-t$, at time t
- $T_0 = 0 < T_1 \dots < T_K$ a set ok $K+1$ successive times and $\tau_k = T_k - T_{k-1}$
- For all $1 \leq k \leq K$, $F_k(t)$ is the discrete forward rate between times T_{k-1} and T_k seen at time $t \leq T_{k-1}$: $\left(F_k(t) = \frac{1}{\tau_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right)\right)$

1.0.2. Diffusion

Under the spot LIBOR probability measure Q^{d} the forward rate diffusion is as follows:

$$\mathrm{dF_k(t)} = \sigma_k(t)F_k(t)\sum_{j=\beta(t)}^{k}\frac{\tau_j\rho_{j,k}\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}\mathrm{dt} + \sigma_k(t)F_k(t)\mathrm{d}Z_k^{\mathrm{d}}(t)$$

With $\rho_{j,k}\mathrm{dt} = \mathrm{d}\langle Z_j, Z_k\rangle_t = \mathrm{d}Z_j(t).\mathrm{d}Z_k(t)$ and Z_k^{d} a M-dimensional (typically M=2) Brownian vector under Q^{d}.

The forward rates modelled here are strictly positive. In the LMM framework, the risk neutral measure is assumed to equal the spot LIBOR measure.

1.1. Displaced Diffusion LMM [2]

1.1.1. Diffusion

Using the same notations as for the LMM above, the forward rate diffusion under the DD-LMM is simply deduced through shifting the forward rates, thanks to a displacement factor δ:

$$\mathrm{dF_k(t)} = (F_k(t)+\boldsymbol{\delta})\left(s(t)^2\sum_{j=\beta(t)}^{k}\frac{\gamma_k(t)\gamma_j(t)\tau_j\left(F_j(t)+\delta\right)}{1+\tau_jF_j(t)}\mathrm{dt} + s(t)\gamma_k(t)\mathrm{d}Z^{\mathrm{d}}(t)\right)$$

1.1.2. Parameter interpretation

Here, the forward rates are shifted so that the quantities $F_k(t)+\delta$ are positive. Finally the rates (forward and zero-coupon rates) are floored at $-\delta$. The shift parameter is therefore easily interpretable and stands for the opposite of the lowest absolute level of the interest rates.

1.2. LMM+ [3]

1.2.1. Diffusion

This model is adapted from the LMM by simultaneously shifting the forward rate diffusion and adding a stochastic variance process. The diffusion is therefore the same as for the DD-LMM, but instead of considering a time-dependent volatility function $s(t)$, a stochastic mean-reversion type Cox-Ingersoll-Ross (CIR) process is used:

$$\begin{cases} \mathrm{d}\left(F_k(t)\right) = (F_k(t)+\delta)V(t)\displaystyle\sum_{j=\beta(t)}^{k}\frac{\gamma_k(t)\gamma_j(t)\tau_j\left(F_j(t)+\delta\right)}{1+\tau_jF_j(t)}\mathrm{dt} + \sqrt{V(t)}\gamma_k(t)(F_k(t)+\delta)\mathrm{d}Z^{\mathrm{d}}(t) \\ \mathrm{d}V(t) = \kappa\left(\theta - V(t)\right)\mathrm{dt} + \epsilon\sqrt{V(t)}\mathrm{d}W(t) \text{ and } \mathrm{d}\langle W, Z^{\mathrm{d}}\rangle_t = \rho\mathrm{dt} \end{cases}$$

1.2.2. Parameter interpretation

Once again, interest rates (forward and zero-coupon rates) are floored at – δ.

The CIR (variance) process parameters are easily interpretable. κ is the mean-reversion coefficient, θ is the long-term variance and ϵ is a long-term volatility coefficient.

2. Gaussian models

The two models presented hereafter focus on the diffusion (assumed Gaussian) of the instantaneous rates, variables that cannot be directly observable on the financial markets contrary to the forward rates.

2.1. 1-factor Hull & White model [4]

2.1.1. Diffusion

The standard 1-factor Hull & White model relies on the following diffusion:

$$dr_t = a[f(t) - r_t]dt + \sigma dz$$

Denoting by r_t the instantaneous short rate, with $r_0 > 0$, $f(t)$ is a deterministic function that enables the model to fit the zero-coupon curve considered as an input.

2.1.2. Parameter interpretation

It is a standard mean-reversion model. The a parameter measures the mean-reversion speed of r_t towards its long-term mean $f(t)$; σ is the long-term volatility.

2.2. G2++ [5]

2.2.1. Diffusion

The short rate dynamic induced by this model is built as the sum of two sub-Hull & White processes with zero long-term mean, $x(t)$ and $y(t)$, and of a scaling function ϕ , chosen so as to fit the zero-coupon rates curve provided as an input. Thus the short rate is given by:
$r(t) = x(t) + y(t) + \phi(t)$ with $r(0) = r_0$
Where processes x and y satisfy:
$dx(t) = -ax(t)dt + \sigma dW_1(t)$ with $x(0) = 0$
$dy(t) = -by(t)dt + \eta dW_2(t)$ with $y(0) = 0$

In practice, (W_1,W_2) is a two-dimensions Brownian motion with instantaneous correlation $dW_1(t)dW_2(t) = \rho dt$ and $-1 \leq \rho \leq 1$

2.2.2. Parameter interpretation

r_0, a, b, σ, η are positive constants so that a and b are mean-reversion coefficients, σ and η are the long-term volatilities of each process.
ϕ is a deterministic function defined on any projection horizon and such that $\phi(0) = r_0$.

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